GIRRAWEEN HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION

2018

MATHEMATICS EXTENSION 1

Time Allowed: Two hours

(Plus 5 minutes reading time)

Instructions To Students

- **Attempt all questions.**
- **All necessary working must be shown for questions 6-10.**
- **Marks may be deducted for careless or badly arranged work.**
- **Board approved calculators may be used.**
- **For Questions 1-5, write the letter corresponding to the correct answer on your answer sheet.**
- **For Questions 6-10, start each question on a new sheet of paper. Each question should be clearly labelled.**
- **Write 'End of Solutions' on your answer paper when you finish answering all questions.**

For Questions 1-5, write the letter corresponding to the correct answer on your answer sheet (5 marks)

1. Let A be the point (-6,4) and B the point (5,1). Find the coordinates of the point which divides AB internally in the ratio 3 : 4.

   (A) $(-39,12)$ (B) $\left(-\frac{12}{7},\frac{23}{7}\right)$ (C) $\left(-\frac{9}{7},\frac{19}{7}\right)$ (D) (-5, 9)

2. What is the acute angle between the lines $3x+4y-12=0$ and $y=2x-1$.

   (A) 77°28' (B) 81°52' (C) 70°32' (D) 79°42'

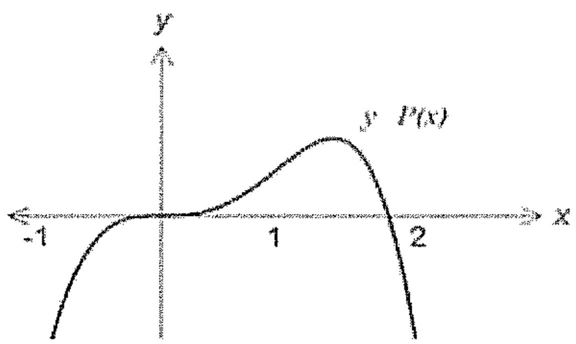
3. At a leadership meeting, two teachers and six students sit at a round table. In how many ways can they be arranged if the two teachers are to be separated?

   (A) 720 (B) 1440 (C) 3600 (D) 5040

4. The coefficient of x^3 in the expansion of $(3-4x)(1+x)^4$.

   (A) -12 (B) 8 (C) 12 (D) 24

5. Which of the following could be the equation of the polynomial $y = P(x)$?



   (A) $P(x)=x^3(2-x)$

   (B) $P(x)=x^2(2-x)^2$

   (C) $P(x)=x^3(x-2)$

   (D) $P(x)=-x^3(x+2)$

Question 6 (19 marks)

(a) Solve $\dfrac{x-2}{x+3} > -2$ 3

(b) Find the coordinates of the point P which divides the interval joining $A(\frac{17}{3},2)$ and $B(-3,4)$ externally in the ratio 2:3. 3

(c) Solve: $\log_{10}(3x-4)+\log_{10}(x-1)=1$ 4

(d) **(i)** Express $\cos\theta-\sqrt{3}\sin\theta$ in the form $R\cos(\theta+\alpha)$ where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$. 3

(ii) Hence solve $\cos\theta-\sqrt{3}\sin\theta=1$, $0\le\theta\le 2\pi$. 3

(e) The gradient of the tangent at any point on a curve is given by $\dfrac{dy}{dx}=\dfrac{2}{2x+1}$. If the curve passes through the point $(1,\log_e 3)$

(i) Find the equation of the curve. 2

(ii) Find the y – intercept of the curve. 1

Question 7(23 marks)

(a) Differentiate:

(i) $y=x^3\log_e(x+1)$ **(ii)** $y=\dfrac{\log_e x}{x^2}$ **(iii)** $y=\log_e\left(\dfrac{x^2}{x+1}\right)$ 9

(b) Find:

(i) $\int (x-1)e^{x^2-2x}\,dx$ **(ii)** $\int \dfrac{x^4}{2x^5+5}\,dx$ **(iii)** $\displaystyle\int_0^{\log_e 7} e^{2x}\,dx$ 9

(c) **(i)** Find the derivative of $y=\log_e(\log_e x)$ 2

(ii) Hence find the exact value of $\displaystyle\int_e^{e^2}\frac{dx}{x\log_e x}$. 3

Question 8 (13 marks)

(a) Use Mathematical Induction to prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 3 for $n \geq 1$. 5

(b) The first three terms in the expansion of $(1+cx)^n$, where n is a positive integer are $1+60x+1680x^2$. Find c and n. 3

(c) A test consists of five multiple choice questions. Each question has five alternative answers. For each question only one of the alternative answers is correct. Katie randomly selects an answer to each of the five questions. What is the probability that Katie selects

(i) all correct answers 1

(ii) three or more correct answers 3

(iii) at least one incorrect answer 1

Question 9 (16 marks)

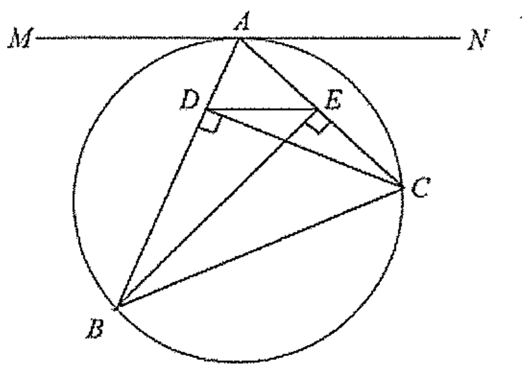
(a) ABC is a triangle inscribed in a circle. MAN is the tangent at A to the circle ABC.

CD and BE are altitudes of the triangle.

(i) Copy the diagram into your answer booklet. 1

(ii) Prove that $BCED$ is a cyclic quadrilateral. 1

(iii) Hence show that DE is parallel to MAN. 3



(b) Solve $\sqrt{3}\sin\theta+\cos\theta=1$ by using $t=\tan\dfrac{\theta}{2}$, $0 \leq \theta \leq 2\pi$. 4

(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points which move on the parabola

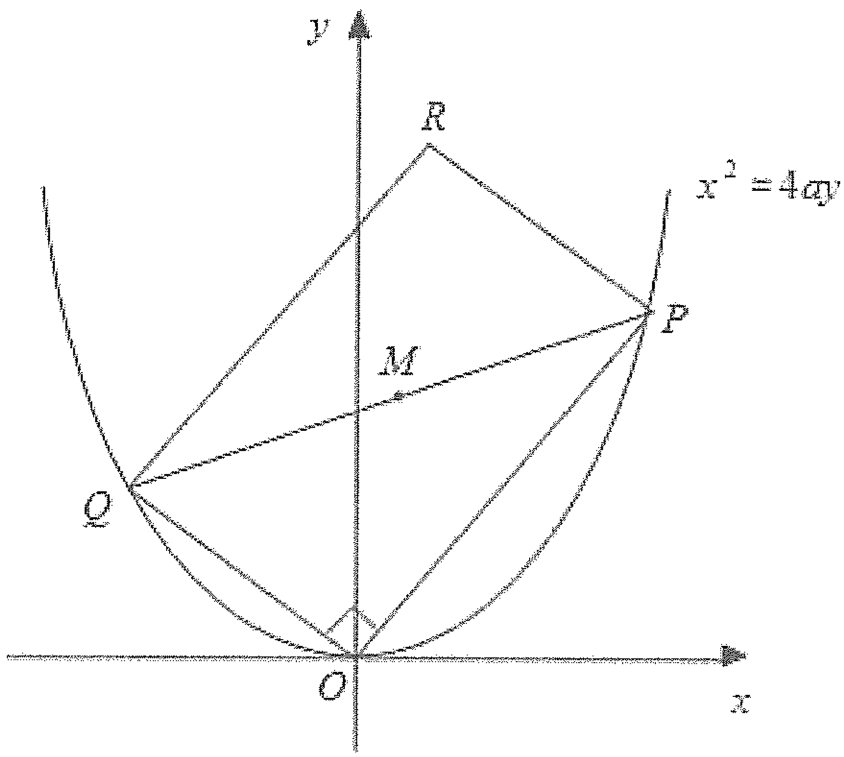
$x^2 = 4ay$ such that $\angle POQ = 90°$, where (0,0) is the origin. $M\left(a(p+q), \dfrac{a(p^2+q^2)}{2}\right)$

is the midpoint of PQ. R is the point such that $OPRQ$ is a rectangle.

(i) Show that $pq = -4$. 2

(ii) Show that R has coordinates $\left(2a(p+q), a(p^2+q^2)\right)$. 2

(iii) Find the equation of the locus of R. 3



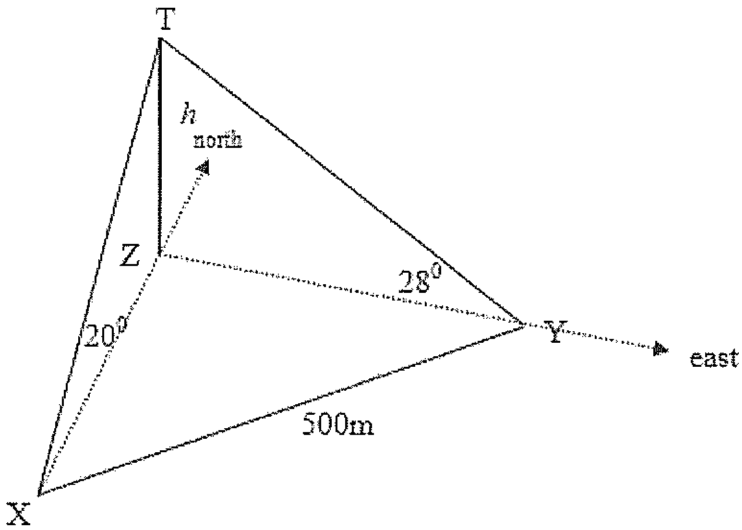
Question 10 (19 marks)

(a) Let $P(x) = (x+1)(x-3)Q(x) + a\ (x+1) + b$, where $Q(x)$ is a polynomial and a and b are real numbers. When $P(x)$ is divided by $x+1$, the remainder is -11and when it is divided by $x-3$, the remainder is 1. Find the remainder when $P(x)$ is divided by $(x+1)(x-3)$. 4

(b) A conservationist observes the angle of elevation of the top of a tree, which is h metres tall, from two positions. From a point X, due south of the tree, it is $20°$ and from point Y, due east of the tree, it is $28°$. The distance XY is 500 metre.

(i) Write expressions for XZ and YZ in terms of h. 4

(ii) Calculate the value of h. 3



(c) Find the area bounded by the curve $y = \log_e x$, the $x-$ axis, the $y-$ axis and the line $y = 2$. Write the answer correct to 2 decimal places. 4

(d) Find the volume of the solid formed when the curve $y = \dfrac{2}{\sqrt{2x-1}}$ is rotated about the $x-$axis from $x = 1$ to $x = 5$, giving an exact answer. 4

End of Examination

Please remember to write 'End of Solutions' on your answer paper.

Year 12 Extension 1 Test 2 (HY) 2018 – Solutions

Multiple choice

1C 2D 3C 4A 5A

Question 6 (19 marks)

(a) $\dfrac{x-2}{x+3} > -2$

$(x+3)^2 \times \dfrac{x-2}{x+3} > -2(x+3)^2, \; x \neq -3$

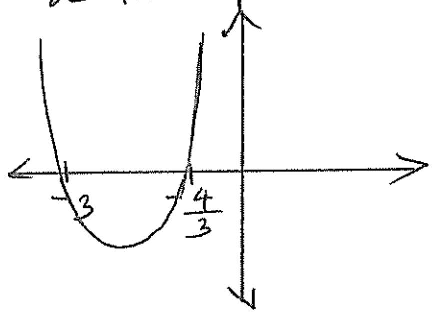
$(x+3)(x-2) > -2(x+3)^2$

$(x+3)(x-2) + 2(x+3)^2 > 0$

$(x+3)\left[x-2+2(x+3)\right] > 0$

$(x+3)(3x+4) > 0$

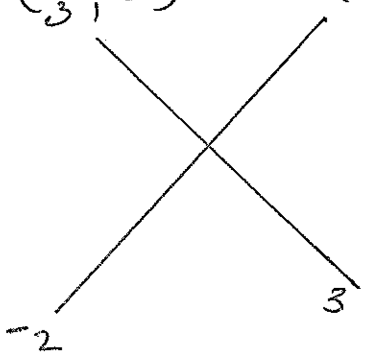
x intercepts $x=-3, \; x = -\dfrac{4}{3}$



3

$\underline{\underline{x < -3 \text{ or } x > -\dfrac{4}{3}}}$

(b) $A\left(\dfrac{17}{3}, 2\right)$ $B(-3, 4)$



$x = \dfrac{(-2 \times -3) + \left(3 \times \frac{17}{3}\right)}{3 + -2} = 23$

$y = \dfrac{(-2 \times 4) + (3 \times 2)}{3 + -2} = -2$

$\underline{\underline{P(23, -2)}}$ 3

(c) $\log_{10}(3x-4) + \log_{10}(x-1) = 1$

$\log_{10}\left[(3x-4)(x-1)\right] = \log_{10} 10$

$(3x-4)(x-1) = 10$

$3x^2 - 7x - 6 = 0$

$pq = -18$

$p+q = -7$

$-9, 2$

$3x^2 - 9x + 2x - 6 = 0$

$3x(x-3) + 2(x-3) = 0$

$(x-3)(3x+2) = 0$

$x = 3$ or $x = \dfrac{-2}{3}$

$\underline{\text{when } x = 3}$

4

$3x - 4 = 9 - 4 > 0$

$x - 1 = 3 - 1 = 2 > 0$

$\underline{\text{when } x = -\frac{2}{3}}$

$3x - 4 = 3 \times \dfrac{-2}{3} - 4 = -6 < 0$

$x - 1 = -\dfrac{2}{3} - 1 < 0$

$x = \dfrac{-2}{3}$ is not a solution.

$\therefore$ $\underline{\underline{\text{the solution is } x = 3}}$

(d)(i) $\cos\theta - \sqrt{3}\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$1 = R\cos\alpha$

$\sqrt{3} = R\sin\alpha$

$R^2 = 4$

$R = 2$ (since $R > 0$)

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\cos\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{3}$ 3

$\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$

(ii) $2\cos\left(\theta + \frac{\pi}{3}\right) = 1$

$\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$

Let $u = \theta + \frac{\pi}{3}$

$\cos u = \frac{1}{2}$ $\quad \frac{\pi}{3} \le \theta + \frac{\pi}{3} \le \frac{7\pi}{3}$

$u = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = 0, \frac{4\pi}{3}, 2\pi$ 3

(e)(i) $y = \int \frac{2}{2x+1}\,dx$

$= \log_e(2x+1) + c$

$\log_e 3 = \log_e 3 + c \quad \therefore c = 0$

$y = \log_e(2x+1)$ 2

(ii) substitute $x = 0$

$y = \log_e 1 = 0$ 1

Question 7 (23 marks)

(a)(i) $y = x^3\log_e(x+1)$

$y' = x^3 \times \frac{1}{x+1} + \log_e(x+1) \times 3x^2$

$= \frac{x^3}{x+1} + 3x^2\log_e(x+1)$ 3

(ii) $y = \frac{\log_e x}{x^2}$

$y' = \frac{x^2 \times \frac{1}{x} - \log_e x \times 2x}{x^4}$ 3

$= \frac{x - 2x\log_e x}{x^4} = \frac{1 - 2\log_e x}{x^3}$

(iii) $y = \log_e\left(\frac{x^2}{x+1}\right)$

$= \log_e x^2 - \log_e(x+1)$

$y' = \frac{1}{x^2} \times 2x - \frac{1}{x+1} = \frac{2}{x} - \frac{1}{x+1}$

$= \frac{x+2}{x(x+1)}$ 3

(b) (i) $\int (x-1)e^{x^2-2x}\,dx$

$= \int \frac{2(x-1)}{2} e^{x^2-2x}\,dx$

$= \frac{1}{2}\int (2x-2)\, e^{x^2-2x}\,dx$

$= \frac{1}{2} e^{x^2-2x} + C$ (3)

(ii) $\int \frac{x^4}{2x^5+5}\,dx$

$= \frac{1}{10}\int \frac{10x^4}{2x^5+5}\,dx$ (3)

$= \frac{1}{10}\log_e(2x^5+5) + C$

(iii) $\int_0^{\log_e 7} e^{2x}\,dx$

$= \left[\frac{e^{2x}}{2}\right]_0^{\log_e 7}$

$= \frac{1}{2}\left(e^{2\log_e 7} - e^0\right)$

$= \frac{1}{2}\left(e^{\log_e 7^2} - 1\right)$ (3)

$= \frac{1}{2} \times 48 = 24$

(c) (i) $y = \log_e(\log_e x)$

$y' = \frac{1}{\log_e x} \times \frac{1}{x} = \frac{1}{x\log_e x}$

(2)

$\int \frac{1}{x\log_e x}\,dx$

$\int \frac{1}{x\log_e x}\,dx = \log_e(\log_e x) + C$

$\int_e^{e^2} \frac{1}{x\log_e x}\,dx = \left[\log_e(\log_e x)\right]_e^{e^2}$

$= \log_e(\log_e e^2) - \log_e(\log_e e)$

$= \log_e(2)$ (3)

Question 8 (13 marks)

$n=1$

$1^3 + (1+1)^3 + (1+2)^3$

$= 1 + 8 + 27 = 36$ which is divisible by 3

Assume true for $n=k$

$k^3 + (k+1)^3 + (k+2)^3 = 3P$ where P is an integer ——— (1)

Aim: to prove true for $n = k+1$

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 3Q$

where Q is an integer ——— (2)

LHS of ② $= (k+1)^3 + (k+2)^3 + (k+3)^3$

$= (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27$

$= 3p + 9k^2 + 27k + 27$

$= 3(p + 3k^2 + 9k + 9)$ (5)

$= 3Q$ where $Q = p + 3k^2 + 9k + 9$ is an integer.

$\therefore$ if the result is true for $n=k$ then it is true for $n=k+1$.

Hence by the principle of Mathematical induction the result is true for all positive integers.

(b) $(1+cx)^n = 1 + {}^nC_1 (cx) + {}^nC_2 (cx)^2 + \cdots$

$= 1 + nCx + \frac{n(n-1)}{2} C^2 x^2 + \cdots$

$nCx = 60x$

$nC = 60$

$C = \frac{60}{n}$

$\frac{n(n-1)C^2}{2} = 1680$

$\frac{n(n-1)}{2} \times \frac{3600}{n^2} = 1680$

$\frac{(n^2-n)1800}{n^2} = 1680$

$1800n^2 - 1800n = 1680n^2$

$120n^2 - 1800n = 0$

$n(120n - 1800) = 0$ (3)

$n = 0$ or $n = 15$

$C = \frac{60}{15} = 4$

(c) $p = \frac{1}{5}$ $q = \frac{4}{5}$

(i) $P(x=5) = {}^5C_5 \left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^0 = \left(\frac{1}{5}\right)^5$ (1)

$= \frac{1}{3125}$

(ii) $P(x=3) + P(x=4) + P(x=5)$

$= {}^5C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2 + {}^5C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^1 + \left(\frac{1}{5}\right)^5$

$= \frac{181}{3125}$ (3)

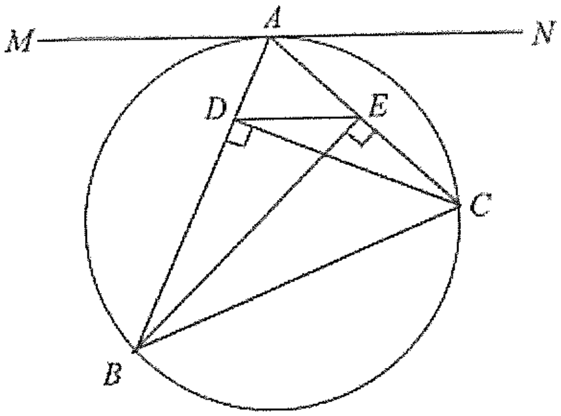
(iii) $P(\text{atleast one incorrect})$

$= 1 - P(\text{all correct})$ ①

$= 1 - \left(\frac{1}{5}\right)^5 = \frac{3124}{3125}$

Question 9 (16 marks)

(a)
(i)



①

(ii) Interval BC subtends equal angles BDC and BEC at points D and E which are on the same side of BC.

$\therefore$ BCED is a cyclic quadrilateral ①

(iii) $\angle ADE = \angle ACB$ (exterior angle of cyclic quadrilateral BCED is equal to opposite interior angle)

$\angle MAB = \angle ACB$ (angle between tangent and chord is equal to angle in the alternate segment)

Hence $\angle MAB = \angle ADE$ ③

$\therefore DE \parallel MN$ (equal alternate angles on transversal AB)

(b) $\sqrt{3}\sin\theta + \cos\theta = 1$

$\sqrt{3} \times \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$

$\frac{2\sqrt{3}t + 1 - t^2}{1+t^2} = 1$

$2\sqrt{3}t + 1 - t^2 = 1 + t^2$

$2t^2 - 2\sqrt{3}t = 0$

$2t(t - \sqrt{3}) = 0$

$t = 0$ or $t = \sqrt{3}$

$\tan\frac{\theta}{2} = 0 \quad 0 \le \theta \le 2\pi$

$\tan u = 0 \quad 0 \le u \le \pi$

$u = 0, \pi$

$\theta = 0, 2\pi$

$\tan\frac{\theta}{2} = \sqrt{3} \quad 0 \le \theta \le 2\pi$

$\tan u = \sqrt{3} \quad 0 \le u \le \pi$

$u = \frac{\pi}{3}$

$\theta = 2u = \frac{2\pi}{3}$ (4)

Check for $\theta = \pi$

LHS $= \sqrt{3}\sin\pi + \cos\pi$

$= \sqrt{3} \times 0 + -1 = -1$

RHS $= 1$

LHS $\neq$ RHS

$\therefore \theta = \pi$ is not a solution

$\theta = 0, \frac{2\pi}{3}, 2\pi$

(c)(i) $O(0,0) \quad P(2ap, ap^2)$

$Q(2aq, aq^2)$

$m_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$

$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$

$m_{OP} \times m_{OQ} = -1$ (2)

$\frac{p}{2} \times \frac{q}{2} = -1$

$\frac{pq}{4} = -1 \quad \therefore pq = -4$

(ii) Let (x, y) be the coordinates of R

$\frac{x}{2} = a(p+q)$

$x = 2a(p+q)$

$\frac{y}{2} = \frac{1}{2}a(p^2+q^2)$ (2)

$y = a(p^2+q^2)$

$\therefore R\big(2a(p+q), a(p^2+q^2)\big)$

(iii) $x = 2a(p+q)$ ① $\quad y = a(p^2+q^2)$ ②

From ① $p+q = \frac{x}{2a}$

From ② $p^2+q^2 = \frac{y}{a}$

$(p+q)^2 = p^2+q^2+2pq$

$$\left(\frac{x}{2a}\right)^2 = \frac{y}{a} + 2\times -4$$

$$\frac{x^2}{4a^2} = \frac{y}{a} - 8$$

$$x^2 = 4a^2 \times \frac{y}{a} - 8 \times 4a^2$$

$$= 4ay - 32a^2$$

$$= \underline{\underline{4a(y-8a)}}$$ (3)

locus is $\underline{\underline{x^2 = 4a(y-8a)}}$

Question 10 (19 marks)

(a) $P(x) = (x+1)(x-3)Q(x) + a(x+1) + b$

Given $P(-1) = -11$

$(-1+1)(-1-3)Q(x) + a(-1+1) + b = -11$

$0 + 0 + b = -11$

$b = -11$

Given $P(3) = 1$

$(3+1)(3-3)Q(x) + a(3+1) + b = 1$

$0 + a(3+1) + b = 1$

$4a - 11 = 1$

$4a = 12$

$a = 3$

Remainder $= a(x+1) + b$ (4)

$= 3(x+1) - 11$

$= \underline{\underline{3x - 8}}$

(b) (i) $\tan 20° = \frac{h}{XZ}$

$XZ = \frac{h}{\tan 20°}$

$\tan 28° = \frac{h}{YZ}$ 4

$YZ = \frac{h}{\tan 28°}$

(ii) $XZ^2 + YZ^2 = 500^2$

$\left(\frac{h^2}{\tan^2 20}\right) + \frac{h^2}{\tan^2 28} = 250000$

$\frac{h^2 \tan^2 28 + h^2 \tan^2 20}{\tan^2 20 \tan^2 28} = 250000$

3

$\frac{h^2(\tan^2 28 + \tan^2 20)}{\tan^2 20 \tan^2 28} = 250000$

$h^2 = \frac{250000 \tan^2 20 \tan^2 28}{\tan^2 28 + \tan^2 20}$

$h = 150\text{m}$

(c) $y = \log_e x$

$x = e^y$

$\text{Area} = \int_0^2 e^y \, dy = [e^y]_0^2$ 4

$= e^2 - 1$

$= 6.39$ square units

(d) $V = \pi \int_1^5 \frac{4}{2x-1} \, dx$

$= \pi \int_1^5 \frac{2 \times 2}{2x-1} \, dx$

$= 2\pi \int_1^5 \frac{2}{2x-1} \, dx$ 4

$= 2\pi \left[\log_e (2x-1)\right]_1^5$

$= 2\pi (\log_e 9 - \log_e 1)$

$= 2\pi \log_e 9$